Computing Klein-Gordon Eigenvalues

Frank Rösler (Cardiff University)

joint work with

Christiane Tretter (University of Bern)

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Introduction

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Solvability Complexity Index (SCI) Hierarchy

General questions:

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$\mathcal{H}$ Hilbert space, $\Omega =$ some class of operators on $\mathcal{H}$.

- Does there exist a sequence (Γ_N) of computer algorithms s.t.

$$\Gamma_N(T) \to \sigma(T) \quad \text{for all } T \in \Omega\,?$$

- Does one have uniform error bounds on Ω?

Introduction

Definition: A *computational (spectral) problem* consists of

- Class of operators Ω,
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- A set Λ of *input information*
  (e.g. point evaluations of PDE coefficients $a \mapsto a(x)$).

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Definition: An *Algorithm* is a map

$$\Gamma : \Omega \to [\text{closed subsets of } \mathbb{C}]$$

such that

- $\Gamma(T)$ depends only on finitely many $f \in \Lambda$,
- $\Gamma(T)$ can be computed using finitely many arithmetic operations on these $f(T)$.

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Example:

- $\Omega = \{H_V = -\Delta + V \text{ on } L^2(\mathbb{R}^d) \,|\, \text{some conditions on } V\}$,
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Can show that[1] if $V \in L^\infty(\mathbb{R}^d;\mathbb{R})$ and $\|V\|_{BV}$ uniformly bounded:

- $\exists$ Algorithms $(\Gamma_n)_{n\in\mathbb{N}}$ s.t. $\Gamma_n(H_V) \to \sigma(H_V)$ (locally in Hausdorff sense) for all V as above,
- explicit error bounds ε_n from below, i.e. $\Gamma_n(H_V) \subset \{z \,|\, \mathrm{dist}(z, \sigma(H_V)) \le \varepsilon_n\}$
- **no** explicit error bounds from above possible.

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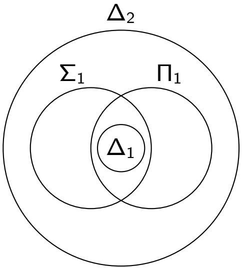
SCI Hierarchy:

$$(\Omega, \Lambda) \in \begin{cases} \Delta_2 & \text{if } \exists \text{ convergent algorithm} \\ \Sigma_1 & \text{if } \in \Delta_2 \text{ and } \exists \text{ error bounds from below} \\ \Pi_1 & \text{if } \in \Delta_2 \text{ and } \exists \text{ error bounds from above} \\ \Delta_1 & \text{if } \exists \text{ error bounds from both sides} \end{cases}$$



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Background

Recent work:

[Hansen(2011)], [Ben-Artzi-Colbrook-Hansen-Nevanlinna-Seidel(2020)]:

- Definition of SCI Hierarchy;
- SCI classification of some (spectral and other) problems;
- wider theory of SCI hierarchy (Π_2, Δ_3, ...).

[Colbrook-Hansen(2020)], [Becker-Hansen(2020)], [Colbrook-Horning-Townsend(2021)]:

- SCI classification for wider classes of spectral problems: computing spectra, spectral measures, spectral gaps, operator semigroups, ...

[Ben-Artzi-Marletta-R.(2020/21)], [R.-Stepanenko(2022)]:

- SCI classification for scattering resonances and Laplacians on domains

[Webb-Olver(2021)]:

- SCI for Jacobi Operators

... and more!

SCI for Klein-Gordon

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Klein-Gordon Eigenvalue Equation:

$$(-\Delta + m^2)u - (V - \lambda)^2 u = 0$$

$\rightsquigarrow$ eigenvalue problem for quadratic operator pencil

$$T_V(\lambda) := -\Delta + m^2 - (V - \lambda)^2$$

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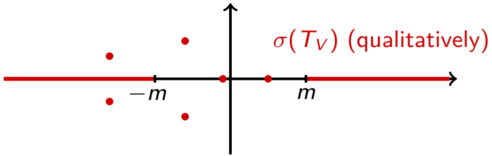

If $|V| \to 0$ at ∞

SCI for Klein-Gordon

Hypothesis:

- $V \in W^{1,p}(\mathbb{R}^d)$ for some $p > d$,
- there exists a constant $M > 0$ such that

$$\|V\|_{W^{1,p}(\mathbb{R}^d)} \leq M, \quad |V(x)| \leq \frac{M}{\sqrt{1+|x|^2}} \quad \text{for all } x \in \mathbb{R}^d.$$

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Theorem (R., Tretter, 2022): There exists a sequence of algorithms Γ_n taking their input from the set of point values of V and producing sets $\Gamma_n(V) \subset \mathbb{C}$ such that

$$d_{\mathrm{H}}(\Gamma_n(V), \sigma_p(T_V)) \to 0 \quad \text{for } n \to \infty.$$

If the constant M is known a-priori, then in addition we obtain the error bound

$$\sup_{z \in \sigma_p(T_V)} \operatorname{dist}(z, \Gamma_n(V)) \leq \frac{1}{n} \quad \text{for all } n \in \mathbb{N}$$

SCI for Klein-Gordon

In the language of the SCI Hierarchy:

- $\Omega := \{V \mid p > d,\ \|V\|_{W^{1,p}} \leq M \text{ and } |V| \leq M(1+|x|^2)^{-\frac{1}{2}}\}$
- $\Lambda := \{T_V \mapsto V(x) \mid x \in \mathbb{Q}^d\}$.

Corollary:

- If M is **not** known a-priori: $(\Omega, \Lambda) \in \Delta_2$,
- If M is known a-priori: $(\Omega, \Lambda) \in \Pi_1$.

SCI for Klein-Gordon

Sketch of proof. Notation: $H_0 = -\Delta + m^2$

- Let $S := VH_0^{-1/2}$ and consider the **compact** operator
$$K(\lambda) := (I - \lambda^2 H_0^{-1})^{-1}\Big(S^*S - \lambda(S^*H_0^{-1/2} + H_0^{-1/2}S)\Big)$$
Then by [Langer-Tretter(2006)] one has
$$\sigma(I - K(\cdot)) \setminus ((-\infty, -m] \cup [m, \infty)) \;=\; \sigma(T_V) \setminus ((-\infty, -m] \cup [m, \infty))$$

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Construct approximation of $K(\lambda)$:

- Choose basis $\{e_i\}_{i\in\mathbb{N}}$ of $L^2(\mathbb{R}^d)$ $\rightsquigarrow$ matrix representation $K_{ij} = \langle e_i, K(\lambda)e_j\rangle_{L^2}$

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$$\langle e_i, (\dots)e_j\rangle_{L^2} = \langle H_0^{-1/2}(I - \bar\lambda^2 H_0^{-1})^{-1}e_i,\, VH_0^{-1/2}e_j\rangle_{L^2}$$
$$= \left\langle \left(\frac{(\xi^2+m^2)^{1/2}}{\xi^2+m^2-\bar\lambda^2}\,\hat e_i\right)^\vee, V\left(\frac{\hat e_j}{\sqrt{\xi^2+m^2}}\right)^\vee \right\rangle_{L^2} \quad \rightsquigarrow \quad \text{quadrature}$$

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- Compute $\|(I - K_N^{\text{trunc}}(\lambda))^{-1}\|_{L^2\to L^2}$ and look for poles. $\rightsquigarrow$ $\Gamma_n(T_V) = \{\text{poles}\}$

Abstract Eigenvalue Enclosures

Bound on non-real eigenvalues:

Proposition (R., Tretter, 2022): If $V \in L^\infty(\mathbb{R}^d)$ and $|V(x)| \to 0$ at $|x| \to \infty$, then

$$\sigma_p(T_V) \setminus \mathbb{R} \subset \left\{ \lambda \in \mathbb{C} \,\middle|\, |\lambda| \leq \sqrt{\|V\|^2_{L^\infty(\mathbb{R}^d)} - m^2},\ \mathrm{Re}(\lambda) \in W(V) \right\},$$

where $W(\cdot) =$ numerical range.

Bound in one space dimension:

Proposition (R., Tretter, 2022): If $d = 1$ and $V \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, then every $\lambda \in \sigma_p(T_V)$ satisfies

$$4|m^2 - \lambda^2| \leq \|V^2 - 2\lambda V\|^2_{L^1(\mathbb{R})}$$

Numerical Results

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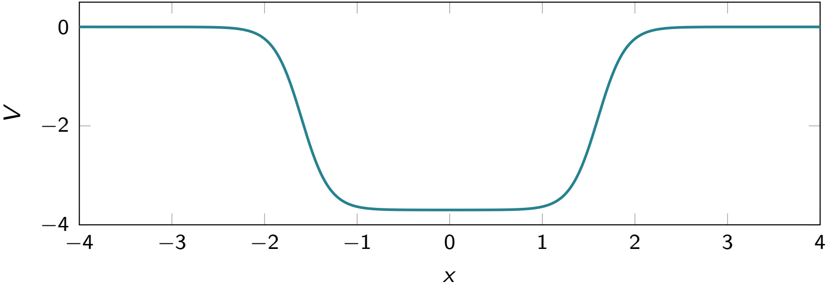
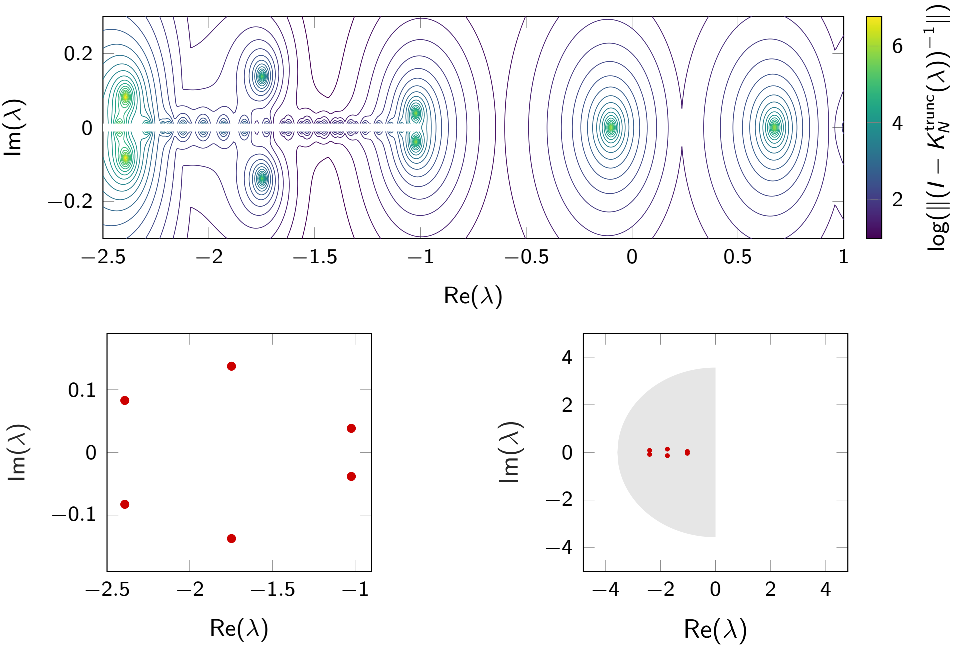
Potential function [Sauter(1931), Lv-Liu-Li-Grobe-Su(2013)]:

$$V(x) = -\frac{v_0}{2}\left(\tanh\left(\frac{x+D/2}{W}\right) - \tanh\left(\frac{x-D/2}{W}\right)\right)$$



$$v_0 = 3.7, \quad D = 3.2, \quad W = 0.3$$

Numerical Results



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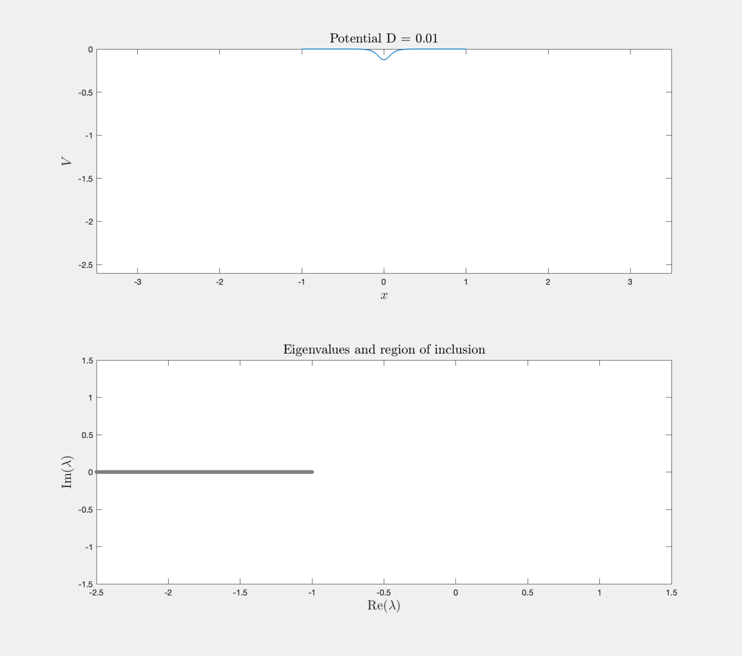

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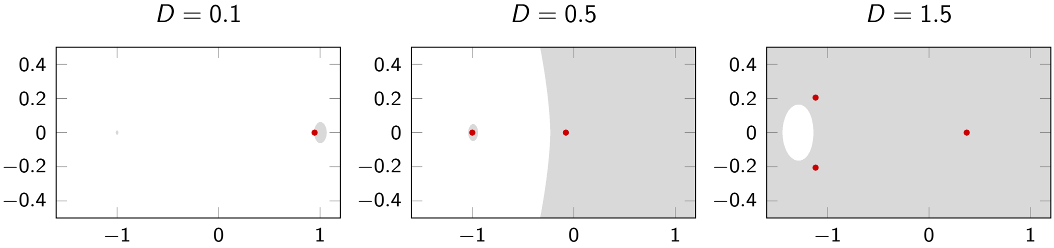

$v_0 = 2.5, \quad W = 0.1$

Thank You!